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APPLICATION OF THE PICARD METHOD OF SEQUENTIAL INTEGRATION IN DIFFERENTIAL INEQUALITIES

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Abstract. In the present paper the connection between differential inequalities and Picard method of the sequential approximations is analyzed. Application of the Chaplygin's theorem in the considered problems is studied.

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1 Introduction

In some cases it becomes very hard to solve differential equations, but we do have a numerical process that can approximate the solution. This process is known as the Picard iterative process (Picard, 1990);(Roose et al., 1995). This method has a large broad of applications in different fields of the investigations of the ordinary differential equations (El-Sayed et al., 2010). This is one of the bases for various iteration methods. We can note the assumptions of the Banach fixed point theorem, the Newton iteration, framed as the fixed point method, demonstrating linear convergence (Vetchinkin, 1935). However, a more detailed analysis shows quadratic convergence under certain circumstances.

First, we consider the relation between the Picard method (the method of sequential integrations) and the S.A. Chaplygin's (Domoshnitskii, 1990) theorem on differential inequalities. For this purpose, we take the following first order equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

with initial condition

$$y_0 = b \; at \; x_0 = a.$$

To get an integral curve at the first approximation take $y = y_0 = const$ in equation (1) and integrate it

$$y_1 = b + \int_a^x f(x,y_0)dx.$$

We note that one can obtain an integral curve of the first approximation in a different way, for example, by specifying its approximate direction from other considerations.

For the second approximation we set

$$y_2' = f(x, y),$$

$$y_2 = b + \int_a^x f(x, y_1)dx = y_1 + \eta_1. \tag{2}$$

Here by η_1 is defined the difference between the first and second approximations.
For the third approximation we have

$$y_3' = f(x, y),$$

$$y_3 = b + \int_a^x f(x, y_1)dx = y_2 + \eta_2, \tag{3}$$

where η_2 stands for the correction for the second approximation for the transition to the third one.

Going the opposite way, we can write

$$\begin{array}{l} y_2' = y_1' + \eta_1', \\ y_3' = y_2' + \eta_2'. \end{array}$$

Further

$$\left.\begin{array}{l} y_2 = y_1 + \int_a^x \eta_1' dx, \\ y_3 = y_2 + \int_a^x \eta_2' dx. \end{array}\right\} \tag{4}$$

It is obvious that if the quantity η_1' keeps its sign (for example positive) in the integral from the point $x_1 = a$ to some point $x_1 = c$. Then in the interval (a, c) will be $y_2 < y_1$. Completely also, if within the interval from a to $x_2 = d$ the correction will probably be $y_3 < y_2$ as in the interval from a to $x_2 = d$.

Let us express the derivatives of the corrections η_1, η_2 through the approximations

$$\left.\begin{array}{l} \eta_1' = y_2' - y_1' = f(x, y_1) - y_1', \\ \eta_2' = y_3' - y_2' = f(x, y_2) - y_2'. \end{array}\right\} \tag{5}$$

Here we specially take both expressions since, by the assumption, the first approximation could be taken quite arbitrarily, and

$$y_1' = f(x, y_0) = f(x, b).$$

Also the second approximation y_2 could be taken quite arbitrary and the choice of the functions $y_1, y_2, ...$ is not restricted any way.

Therefore we can take any function $y_i = \varphi(x)$ and then by the Picard method find the next approximations and state that in the integral from $x_0 = a$ to $x_1 = c$.

$$y_{i+1} \gtrless y_i \ if \ \eta_i = f\left[x, \varphi(x)\right] - \varphi'(x) \gtrless 0. \tag{6}$$

Then after the quantity η_i' changes its sign, the quantity $(y_{i+1} - y_i)$ also could change its sign, but it is not necessary.

Thus, the difference between two neighbor approximations by the Picard method

$$\eta_i = y_{i+1} - y_i = \int_a^x \left\{ f\left[x, \varphi(x)\right] - \varphi'(x) \right\} dx \tag{7}$$

keeps its sign if the integrant keeps its sign calculated by the pure algebraic way

S.A. Chaplygin, starting from general considerations on differential equations (Domoshnitskii, 1990), proved the following important theorem: Let $y = F(x)$ be a solution of differential equation (1). We take another function $z = \varphi(x)$ and write the expression

$$\xi' = f\left[x, \varphi(x)\right] - \varphi'(x). \tag{8}$$

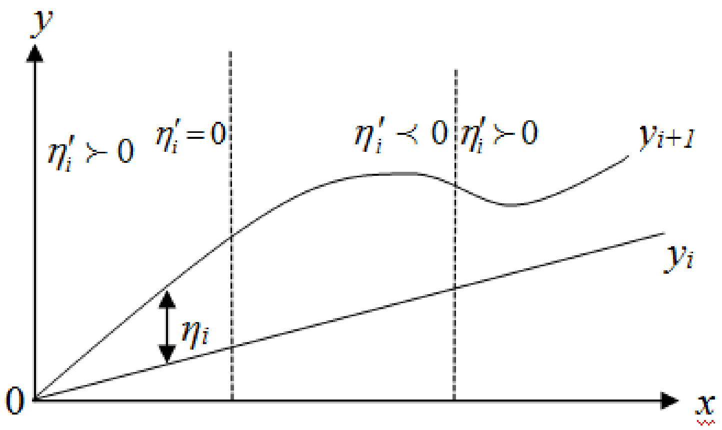

Fig.1.

Then the function ξ' would not change its sign while the difference

$$y - z = F(x) - \varphi(x)$$

keeps its sign same with the sign of ξ' i.e.

$$\left.\begin{array}{l} y > z \ \ if\, f(x,z) - \frac{dz}{dx} > 0, \\ C < z \ \ if \ f(x,z) - \frac{dz}{dx} < 0. \end{array}\right\} \tag{9}$$

Passing to the Picard method we can write the following approximation

$$z_1 = z + \xi = z + \int_a^x \left[f(x,z) - \frac{dz}{dx} \right] dx.$$

Here the theorem on signs is obvious.

Relation (9) differs essentially from (6) by the fact that it contains the difference between exact solution of the differential equation y=$F(x)$ and its approximate solution z=$\varphi(x)$. Using the Picard's method we can estimate the sign only two neibour approximations not known how far removed from the exact solution $F(x)$.

If in the differential equation (1) is valid $\frac{df}{dy} > 0$, then, by the Picard method, all successive approximations will lie on one side of the exact solution if the first approximation lies entirely on one side of it, and if the successive integrations are completed to the end (with the approximate calculation of the integrals, since the discarded terms, however small they may be, can change the sign of the small difference between two neighboring approximations).

Conversely, if $\frac{df}{dy} < 0$ then successive approximations lie on the opposite sides of the exact solution, if the first approximation lies entirely on one side of it.

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